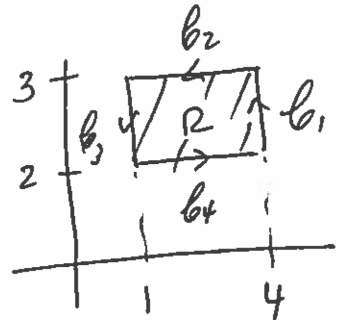
16.4 GREEN'S THEOREM

Start writing on board the following questions:

1) $\displaystyle\int_{\mathcal{C}} 3y\,dx + 5x\,dy$, $\quad \mathcal{C}: x^2+y^2=1.$

2) $\displaystyle\int_{\mathcal{C}} (e^{x^3}+y)\,dx + (x^2+\arcsin y^2)\,dy$

where $\mathcal{C}$:



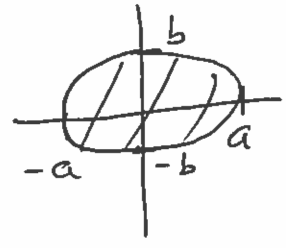
3) $\displaystyle\int_{\mathcal{C}} e^x \sin y\,dx + e^x \cos y\,dy$

Notice:
$Q_x = e^x \cos y = P_y$
$\Rightarrow \vec{F}$ conservative

$\mathcal{C}$:

$\mathcal{C}_1: x^2+y^2=1, \quad y \geq 0$
$\mathcal{C}_2: y=0, \quad -1 \leq x \leq 1.$

4) Find area enclosed by
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



$g_{D'}$

5) $$I = \oint_{\mathcal{C}_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

where $\mathcal{C}_1:\ x^2+y^2=1 \longrightarrow \begin{cases} x(t)=\cos t \\ y(t)=\sin t \end{cases},\quad 0 \le t \le 2\pi.$

answ:

$$I = \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t)\,dt + \int_0^{2\pi} \frac{\cos t}{1}\cos t\,dt$$

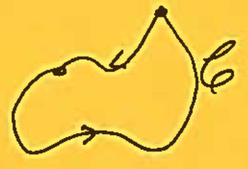
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} dt = \underline{\underline{2\pi}}$$

13. Same as 16.4 Stewart.

Green's Theorem.

Some curves definitions: (xy-plane)

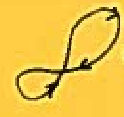
Simple closed:



It doesn't cross itself.

Not Simple:

Not closed



closed. but not simple

Consider the region R bounded by the curve $\mathcal{C}$ (All points inside $\mathcal{C}$ and points on $\mathcal{C}$)

Positive orientation:

the curve $\mathcal{C}$, $\vec{r}(t)$ is traversed counterclockwise as t increases. Then, the region R is to the left of an observer moving along $\mathcal{C}$.

Notation: $\oint_{\mathcal{C}}$ or simply $\oint$.

Thm.- ① $\mathcal{C}$ positively oriented, piecewise smooth, simple closed in $\mathbb{R}^2$ and R bounded by $\mathcal{C}$

② $P(x,y)$, $Q(x,y)$ have continuous partial derivatives on an open region that contains R, then

$$\oint_{\mathcal{C}} \left[P(x,y)\,dx + Q(x,y)\,dy\right] = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

Applications of Green's theorem

Example 1. (Not in book)
$$I = \oint_{\mathscr{C}} 3y\,dx + 5x\,dy$$

where $\mathscr{C}: x^2+y^2=1 \longrightarrow \begin{cases} x(t)=\cos t \\ y(t)=\sin t \end{cases}$ $\quad 0 \le t \le 2\pi$

Answ: 1) Is this a conservative vector field?

$P(x,y)=3y, \qquad Q(x,y)=5x$

$P_y = 3 \neq 5 = Q_x$ It's not conservative

DIRECT INTEGRATION

$$I = \int_0^{2\pi} [3y(t)\,x'(t) + 5x(t)\,y'(t)]\,dt$$
$$= \int_0^{2\pi} [3\sin t(-\sin t) + 5\cos t \cos t]\,dt$$
$$= -3\int_0^{2\pi} \sin^2 t\,dt + 5\int_0^{2\pi} \cos^2 t\,dt = -3\pi + 5\pi = 2\pi$$

$\int \sin^2 t\,dt = \int \frac{1-\cos 2t}{2}\,dt = \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]$

$\int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]$

APPLYING GREEN'S THEOREM

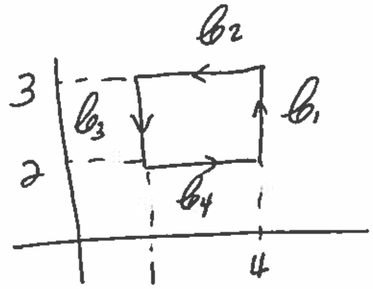
$$\oint_{\mathscr{C}} \overset{=P}{3y}\,dx + \overset{=Q}{5x}\,dy = \iint_{\substack{R \\ x^2+y^2\le 1}} [\overset{Q_x - P_y}{5-3}]\,dx\,dy = 2\iint_{\substack{R \\ x^2+y^2\le 1}} dx\,dy = 2\pi(1)^2 = 2\pi$$

= Area of circle of radius 1

Example 2

Evaluate $I = \oint_{\mathcal{C}} \underbrace{(e^{x^3} + y)}_{=P} dx + \underbrace{(x^2 + \arcsin y^2)}_{=Q} dy$

where $\mathcal{C}$:



Direct integration will be extremely difficult

Using Green's theorem extremely simple.

$$I = \iint_R \overbrace{(2x-1)}^{Q_x - P_y} \, dx\, dy = \int_1^4 \int_2^3 (2x-1)\, dy\, dx =$$

$$= \int_1^4 (2xy - y)\Big|_2^3 dx = \int_1^4 [6x - 3 - (4x-2)]\, dx$$

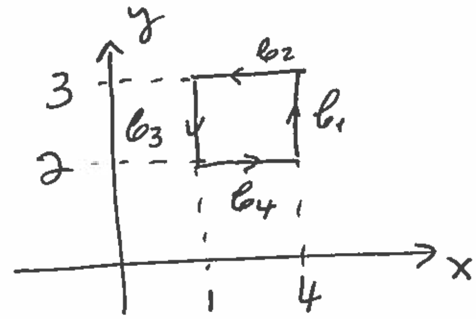
$$= \int_1^4 (2x-1)\, dx = x^2 - x \Big|_1^4 = 16 - 4 - (1-1) = 12$$

What would had happened if $\vec{F} = \langle P, Q \rangle$ were conservative?

Example 2

$$I = \oint_{\mathcal{C}} \overbrace{(e^{x^3}+y)}^{=P(x,y)}\,dx + \overbrace{(x^2 + \arcsin y^2)}^{=Q(x,y)}\,dy$$

directly. where $\mathcal{C}$:



Along $\mathcal{C}_1$: $\vec{r}_1(t) = \langle 4, t\rangle$, $2 \le t \le 3$

$$I = \int_2^3 (e^{64}+t)\,\underbrace{x'(t)}_{=0}\,dt + \int_2^3 (16 + \arcsin t^2)\underbrace{(1)}_{=y'(t)}\,dt$$

Hard!

Along $\mathcal{C}_2$: $\vec{r}_2(t) = \langle 4-t, 3\rangle$, $0 \le t \le 3$

$$I = \int_2^3 \left(e^{(4-t)^3} + 3\right)\underbrace{(-1)}_{=x'(t)}\,dt + 0,$$

Hard!

APPLICATION OF GREEN'S THM : COMPUTING AREAS

Given a region R

$$\text{Area}(R) = \iint_R dA = \iint_R \left[Q_x - P_y\right] dA$$

If $\quad \boxed{Q_x - P_y = 1}$

Some possibilities are:

I $\quad P(x,y) = 0$, $\quad Q(x,y) = x$,

II $\quad P(x,y) = -y$, $\quad Q(x,y) = 0$,

III $\quad P(x,y) = -\frac{1}{2}y$, $\quad Q(x,y) = \frac{1}{2}x$

Then,

$$\text{Area}(R) = \oint_C x\,dy, \quad \text{Area}(R) = \oint (-y)\,dx, \quad \text{Area}(R) = \frac{1}{2}\oint x\,dy - y\,dx$$

Example
(In book)

Find the area enclosed by an ellipse

$$\mathcal{C}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

parametric equations:

$$\mathcal{C}: \vec{r}(t) = \langle a\cos t, b\sin t \rangle = \langle x(t), y(t) \rangle \quad 0 \le t \le 2\pi$$

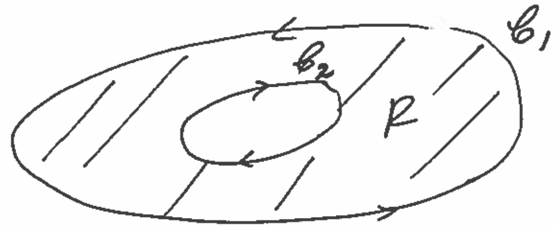
$$\text{Area}(R) = \iint_R dA = \frac{1}{2} \oint x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos t\,(b\cos t)\,dt - b\sin t\,(-a\sin t)\,dt$$

$$= \frac{ab}{2} \int_0^{2\pi} [\cos^2 t + \sin^2 t]\,dt = \frac{ab}{2}(2\pi) = \underline{\underline{ab\pi}}.$$

Extension of Green's theorem to regions with holes.

Consider



Obviously,

R is on the left of an observer moving along $\mathscr{C}_1$ and $\mathscr{C}_2$ according to the orientation shown in the figure: $\mathscr{C}_1$ is positively oriented (Counterclockwise)
$\mathscr{C}_2$ is negatively oriented. (clockwise)

Corollary (Green's Thm for multiply connected regions)

Hyp: a) $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$, $\mathscr{C}_1$ and $\mathscr{C}_2$ Simply closed and piecewise smooth.

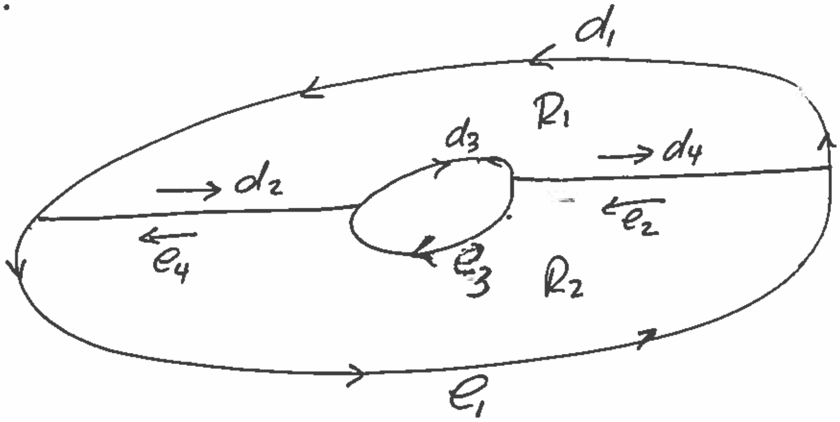
b) $\mathscr{C}_1$ and $\mathscr{C}_2$ enclosing a region R as shown in graph above. $\mathscr{C}_1$ positively oriented and $\mathscr{C}_2$ negatively oriented.

c) P, Q have continuous partial derivatives in open region D s.t $R \subset D$

then,

$$\boxed{\iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \oint_{\mathscr{C}_1} P\,dx + Q\,dy + \oint_{\mathscr{C}_2} P\,dx + Q\,dy}$$

Proof:-



Let $\bar{\mathscr{C}}_1 = e_1 \cup e_2 \cup e_3 \cup e_4$, oriented counterclockwise enclosing R_2

$\bar{\mathscr{C}}_2 = d_1 \cup d_2 \cup d_3 \cup d_4$ oriented counterclockwise enclosing R_1

Applying Green's theorem for simply-connected regions

$$\iint_{R = R_1 \cup R_2} [Q_x - P_y]\, dA = \iint_{R_1} [Q_x - P_y]\, dA + \iint_{R_2} [Q_x - P_y]\, dA$$

$$\overset{\text{Green's Thm}}{=} \oint_{\mathscr{C}_1} P\,dx + Q\,dy + \oint_{\mathscr{C}_2} P\,dx + Q\,dy$$

$$= \oint_{e_1 \cup e_2 \cup e_3 \cup e_4} P\,dx + Q\,dy + \oint_{d_1 \cup d_2 \cup d_3 \cup d_4} P\,dx + Q\,dy$$

$$= \int_{e_1} + \cancel{\int_{e_2}} + \int_{e_3} + \cancel{\int_{e_4}}$$

$$+ \int_{d_1} + \cancel{\int_{d_2}} + \int_{d_3} + \cancel{\int_{d_4}}$$

$$= \int_{e_1} + \int_{e_3} + \int_{d_1} + \int_{d_3}$$

$$= \left(\int_{e_1} + \int_{d_1}\right) + \left(\int_{e_3} + \int_{d_3}\right) =$$

$$= \oint_{\underbrace{e_1 \cup d_1}_{= \mathcal{C}_1}} P\,dx + Q\,dy + \oint_{\underbrace{e_3 \cup d_3}_{= \mathcal{C}_2}} P\,dx + Q\,dy \quad \checkmark$$

<u>Corollary</u>

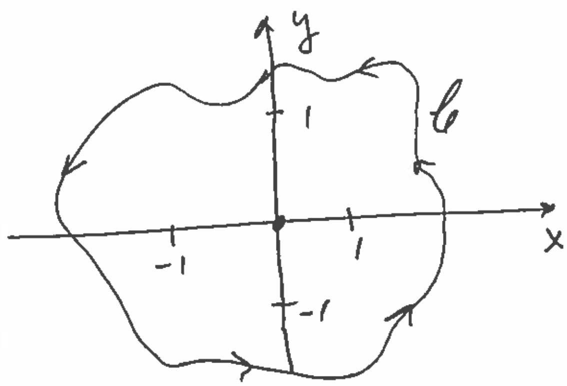
If R is region defined above enclosed by $\mathcal{C}_1$ and $\mathcal{C}_2$ and P and Q have partial derivatives on an open region containing R, then if $\boxed{Q_x = P_y \text{ in } R}$

$$\boxed{\oint_{\mathcal{C}_1} P\,dx + Q\,dy = \oint_{\mathcal{C}_2} P\,dx + Q\,dy}$$

Example (in book) Find $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r}$

where $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle = \langle P, Q \rangle$

and $\mathcal{C}$ is an arbitrary simple closed curve enclosing the origin as in the following graph:

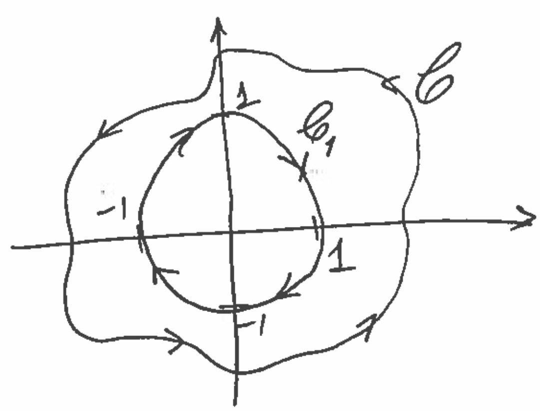


$\mathcal{C}$ is positively oriented.

Notice, $P(x,y)$ and $Q(x,y)$ are not defined at $(0,0)$

So Green's thm does not apply to the whole region R.

However, if we consider the region S enclosed by $\mathcal{C}$ and $\mathcal{C}_1$: $x^2+y^2=1$.



Then P, Q are continuous in S

$$Q_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = P_y$$

Therefore, all conditions of previous Corollary are satisfied and as a consequence

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \oint_{\substack{\mathcal{C}_1 \\ x^2+y^2=1}} P\,dx + Q\,dy$$

Now, $\displaystyle\oint_{\mathcal{C}_1} P\,dx + Q\,dy = \oint_{\mathcal{C}_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

$$= \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t)\,dt + \frac{\cos t}{1}\cos t\,dt = \int_0^{2\pi} dt = \underline{\underline{2\pi}}$$

In conclusion, for any closed curve enclosing the origin and the circle of radius 1 as shown in graph

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \underline{\underline{2\pi}}$$